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A Simplified Adaptive Mesh Technique Derived from the Moving Finite Element Method*

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An adaptive mesh technique is derived as a constrained minimization of the functional used in the moving finite element (MFE) method. The result is a matrix equation for mesh velocities alone (i.e., a pure adaptive mesh prescription). The method retains many attractive features of MFE, including the ability to independently control mesh motion through penalty terms. © 1984 Academic Press, Inc.

A. INTRODUCTION

Adaptive mesh techniques for the solution of partial differential equations have long been of interest, particularly for those problems involving the propagation of sharp fronts through the mesh, such as shocks [1] or flame fronts [2], for example. In general, an adaptive mesh technique may be defined as one which changes the mesh in the course of the calculation in order to improve the accuracy of the solution. Typically, this has meant that the mesh is concentrated in regions of rapidly changing gradients, and various algorithms have been proposed for this purpose (Refs. [1–4], for example).

Among this collection of methods the moving finite element (MFE) method [5–7] stands out because it is directly based on the minimization of a measure of the error of the solution, namely, the L_2 norm of the residual. The method has demonstrated some spectacular results in resolving one-dimensional fronts [7], and there is good reason to believe that it will also be successful in 2-D [8]. The MFE method, in effect, combines the Galerkin finite element method with an adaptive mesh scheme in a self-consistent manner. The good accuracy obtained in the demonstration problems [7] must be attributed to the presence of the adaptive mesh rather than to the use of Galerkin finite elements. Unfortunately, the adaptive method (i.e., the specification of mesh velocities) is intimately interconnected with the discrete approximation for the partial differential equations to be solved. As a result of this coupling the nature of the underlying adaptive scheme is obscured. In addition, this coupling leads to a

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much larger matrix system of equations to be solved, as compared to the system associated with the Galerkin equations alone.

It will be the primary purpose of this paper to show that it is possible to extract this underlying adaptive mesh technique from the MFE method. Given any specified discrete approximation for the partial differential equations, the result will be a matrix system of equations for the mesh velocities alone (i.e., a pure adaptive mesh scheme). This system of equations arises, as in the MFE method, from a variational principle involving the constrained minimization of a functional given by the L_2 norm of a residual. This means that the resulting matrix is sparse, symmetric, and positive definite. Additional control over the motion of mesh points can be exercised, as in the MFE method, by the addition of "penalty" or "regularization" terms to the functional.

Just as in the case of the MFE method, the new technique is derived in general terms, but it is applied and discussed in a rather simple context, namely the 1-D Burgers equation, for clarity of exposition. The method shares many characteristics in common with MFE and these are generally not discussed in order to avoid unnecessary repetition. These have to do principally with applications to systems of equations and with the issues involved in the choice of penalty functions. It is expected that any such procedures applied to the MFE method will apply equally well to this simplified method.

B. General Description

1. *The Moving Finite Element Method (MFE)*

We begin with a short recapitulation of the essentials at the MFE method as it applies in 1-D [7]. Consider the scalar partial differential equation

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + L(q) = 0, \tag{1}$$

where q is an arbitrary scalar dependent variable, u is a velocity, and L is some possibly nonlinear spatial differential operator. This equation is a typical transport equation of interest in many applications. We wish to solve this equation for $t \geqslant 0$ on the interval $x \in [a, b]$ given initial and boundary condition. Assume a regular subdivision of this interval (mesh) defined by $[a = x_1 < x_2 \cdots x_i < x_{i+1} \cdots < x_N = b]$. We now assume that the mesh can move with an arbitrary velocity S ($S_i \equiv dx_i/dt$). Recalling that the time derivative following the motion of the mesh is

$$\dot{q}\left(\equiv \frac{dq}{dt}\right) = \frac{\partial q}{\partial t} + S\frac{\partial q}{\partial x},$$

the transport equation can be written as

$$\dot{q} + (u - S)\frac{\partial q}{\partial x} + L(q) = 0. \tag{2}$$

Let us consider the subinterval $x^k: x \in [x_i, x_{i+1}]$, $1 \leqslant k \leqslant N-1$. In each subinterval we assume a linear variation[1] for our variables

$$q^k(x) = \sum_{j=i}^{i+1} q_j \phi_j^k(x), \qquad \dot{q}^k(x) = \sum_{j=i}^{i+1} \dot{q}_j \phi_j^k(x),$$

$$u^k(x) = \sum_{j=i}^{i+1} u_j \phi_j^k(x), \qquad S^k(x) = \sum_{j=i}^{i+1} S_j \phi_j^k(x),$$

where $q_j \equiv q(x_j)$, etc., and ϕ_i^k, ϕ_{i+1}^k are the linear *shape* functions (to be distinguished from the related α_i, β_i *basis* functions of Ref. [7]) on the interval k defined by

$$\left.\begin{aligned} \phi_i^k(x) &= \frac{x - x_{i+1}}{x_i - x_{i+1}} \\ \phi_{i+1}^k(x) &= \frac{x - x_i}{x_{i+1} - x_i} \end{aligned}\right\} x \in [x_i, x_{i+1}], \tag{3}$$

and zero otherwise. Note that $\phi_i^k(x_i) = 1$, $\phi_i^k(x_{i+1}) = 0$, and $\phi_{i+1}^k(x_i) = 0$, $\phi_{i+1}^k(x_{i+1}) = 1$. Since we have assumed a linear variation for the variables, the transport equation, Eq. (2), will not be exactly satisfied and there will be a nonzero residual on the interval given by

$$R^k = \sum_{j=i}^{i+1} \left[\dot{q}_j + (u_j - S_j) \frac{\partial q^k}{\partial x} \right] \phi_j^k(x) + L(q^k), \tag{4}$$

where $\partial q^k / \partial x = (q_{i+1} - q_i)/(x_{i+1} - x_i)$. To determine the unknowns $\dot{q}_i$, S_i, $1 \leqslant i \leqslant N$, we minimize a functional I, equal to the square of the norm of the residual,

$$I = \|R\|^2 = \int_{x_1}^{x_N} R^2 \, dx = \sum_k \int_{x_i}^{x_{i+1}} (R^k)^2 \, dx,$$

and obtain the equations to be solved by differentiating with respect to the free parameters $\dot{q}_i$, S_i,

$$\frac{\partial I}{\partial \dot{X}_j} = 0, \qquad \dot{X}_j \in \{\dot{q}_i, S_i\}, \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant 2N,$$

and since the square of the norm is quadratic in the variables, this defines a system of $2N$ linear equations which is expressed by the matrix equation

$$A\dot{\mathbf{X}} = \mathbf{b}, \tag{5}$$

where we have defined a vector, $\dot{\mathbf{X}}^T = (\dot{\mathbf{q}}^T, \mathbf{S}^T)$, composed of the $2N$ unknown parameters $\dot{X}_j$. This equation can be viewed as a linear system of ODEs to be solved

[1] The MFE method [7] is more general, allowing arbitrary basis functions, but the linear case is the most practical.

for the associated vector $\mathbf{X}^T = (\mathbf{q}^T, \mathbf{x}^T)$ as a function of time. Standard methods, in particular stiff ODE solvers, are applicable.

Notice that in the special case of a simple transport equation $(L(q) \equiv 0)$, the residual (Eq. (4)) is made identically zero by $\dot{q}_i = 0$, $S_i = u_i$, $1 \leqslant i \leqslant N$, and so this provides a simple exact solution of Eq. (5).

Under certain circumstances the matrix A of Eq. (5) can become singular. This occurs whenever $\partial q^{k-1}/\partial x = \partial q^k/\partial x$ (i.e., whenever q_{i-1}, q_i, q_{i+1} lie along a single straight line [5]). The explanation for this is that the unknowns $\dot{q}_i$, S_i occur in the linear combinations $\dot{q}_i + (u_i - S_i)\,\partial x$, and there are exactly $2N$ of these provided $\partial q^{k-1}/\partial x \neq \partial q^k/\partial x$. However, if these slopes are equal there are only $2N-1$ linearly independent combinations and the matrix A becomes rank deficient, i.e., there are fewer independent equations than unknowns. The practical effect is that the mesh velocity S_i becomes extremely large as this condition of colinearity is approached. This can be intuitively understood since a mesh point located in the interior of a linear segment of the solution is not needed to resolve that segment, and it thus tends to move rapidly towards one or the other end of the segment. Thus we would normally not expect this behavior to occur in the course of a solution, but it may easily arise as a result of improper initial mesh placement. To overcome such potential problems, and to provide independent control over mesh velocities, Miller [5, 6] introduced regularization or penalty terms which are added to the square of the norm of the residual to obtain a new functional

$$I \equiv \|R\|^2 + \sum_k P_k^2(S_i), \tag{6}$$

which is again to be minimized with respect to $\dot{q}_i$, S_i. Notice that the penalty functions $P_k(S_i)$ are functions of the mesh velocities only, so that they affect only the mesh velocity equations. The functional I remains positive definite, and hence leads to a symmetric, positive definite system of equations.

2. *The Simplified Moving Finite Element Method (SMFE)*

Examining Eq. (5) we see that A is a $2N \times 2N$ matrix which may be partitioned as

$$A = \left[\begin{array}{c|c} A_1 & B \\ \hline B^T & A_2 \end{array}\right], \tag{7}$$

where A_1, A_2, and B are $N \times N$ submatrices, corresponding to the partitioning of the vector $\dot{\mathbf{X}}$

$$\dot{\mathbf{X}}^T = (\dot{\mathbf{q}}^T \,|\, \mathbf{S}^T). \tag{8}$$

This is a natural partitioning into the two physically different quantities $\dot{q}$ and S. The matrices A_1 and A_2 can be considered to be associated with the constrained minimization problems

$$\left.\begin{array}{ll} I_1 = I(\mathbf{S}=0), & \dfrac{\partial I_1}{\partial \dot{q}_j} = 0, \\ I_2 = I(\dot{\mathbf{q}}=0), & \dfrac{\partial I_2}{\partial S_j} = 0, \end{array}\right\} 1 \leqslant j \leqslant N \tag{9}$$

respectively. The matrix B is a coupling matrix. This coupling is inherent in the MFE method, and it may be viewed as a drawback in the sense that it leads to very large matrix equations. This may not seem significant for the scalar equation considered here, but it can become a severe problem for systems of equations, where all the variables become coupled.

We can reduce the size of this system of equations by enforcing certain constraints, as in Eqs. (9). For example, if we enforce the constraint $\mathbf{S} = 0$, i.e., a fixed mesh, then the method reduces to just the usual Galerkin finite element method expressed as

$$A_1 \dot{\mathbf{q}} = \mathbf{b}_1, \tag{10}$$

which is obtained from the first of Eq. (9). On the other hand, if we enforce the constraint $\dot{\mathbf{q}} = 0$ then the method becomes a finite element version of Harlow's dynamics of contours method [9]. This last scheme, expressed as

$$A_2 \mathbf{S} = \mathbf{b}_2, \tag{11}$$

and obtained from the last of Eq. (9), is a viable adaptive mesh scheme except under conditions when the solution tends to generate new contour values. Notice that these constraints are applied to the functional I, rather than to the matrix equation (Eq. 5).

These ideas can be generalized. Suppose we already have a convenient discrete approximation to Eq. (2) in the form

$$\dot{q}_i = Q_i(\mathbf{S}, \mathbf{q}, \mathbf{x}), \tag{12}$$

defined on the mesh $\mathbf{x}^T = \{x_i\}$. This may be a finite-difference approximation, for example, or any one of other reasonable approximations such as appear in numerous existing codes. Equations (12) can of course be integrated to find $\mathbf{q}$, provided $\mathbf{S}$ is known. In order to find an equation for $\mathbf{S}$, Eq. (12) may be viewed as a *prescribed relationship* between $\dot{\mathbf{q}}$ and $\mathbf{S}$ which acts as a constraint to the minimization of the functional I. Using Eqs. (12) to eliminate $\dot{\mathbf{q}}$ in Eq. (4), we obtain

$$R^k = \sum_{j=i}^{i+1} \left[Q_j(\mathbf{S}, \mathbf{q}, \mathbf{x}) + (u_j - S_j) \frac{\partial q^k}{\partial x} \right] \phi_j^k(x) + L(q^k), \tag{13}$$

which is a function of S_j only. Substituting this into the expression for the functional as defined in Eq. (6), we perform the minimization

$$\frac{\partial I}{\partial S_i} = 0, \qquad 1 \leqslant i \leqslant N,$$

which implies a matrix equation

$$A_3 \mathbf{S} = \mathbf{b}_3, \tag{14}$$

where A_3 is an $N \times N$ matrix. The procedure to obtain Eq. (14) is illustrated by a concrete example in Appendix A.

The above procedure defines the simplified moving finite element method (SMFE). The method will result in the set of equations

$$A_3 \mathbf{S} = \mathbf{b}_3. \tag{15a}$$

$$\dot{\mathbf{q}} = \mathbf{Q}(\mathbf{S}, \mathbf{q}, \mathbf{x}), \tag{15b}$$

$$\dot{\mathbf{x}} = \mathbf{S}. \tag{15c}$$

In this form, the method resembles a conventional adaptive mesh scheme; it contains an adaptive mesh-motion algorithm (Eqs. (15a), (c)) together with discrete evolution equations for the dependent variable $\mathbf{q}$ on this moving mesh (Eq. (15b)). We have thus, in a sense, extracted the adaptive mesh scheme contained in the MFE method. It is easy to show that the new matrix A_3 remains sparse, symmetric, and positive definite. We have also retained, as a convenience, the ability to independently control the mesh motion through penalty functions (regularization terms) added to the functional I, even though the new matrix is not necessarily subject to the same singularities as the previous one. The important point to note is that the size of the linear matrix problem has been greatly reduced, especially for problems involving systems of equations. This is very significant when Eqs. (15b), (c) are solved using explicit methods, but less significant when stiff equation solvers are used.

It is not clear what we have sacrificed in comparison with MFE. There is great freedom in the choice of the discrete approximation, Eq. (12). It may be expected that the mesh motion will be sensitive to the choice of discretization, but since the method reduces the global error of the solution it should tend to compensate for the discretization error of the chosen scheme. We might expect that not all discretizations will be successful, especially in cases of large and abrupt changes in mesh size, since in such a case the discretization error will be especially large, and it may not be possible for the mesh motion alone to overcome this large error. These considerations will be illustrated when numerical examples are presented in the following section.

C. The Test Problem

To illustrate the properties of this new adaptive mesh scheme we consider the one-dimensional Burgers equation, which is a simple example of Eq. (1). This equation has been previously used to illustrate the MFE method [5–7] since it is one of the simplest equations which exhibit traveling front solutions. Such a traveling front solution of the Burgers equation,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{1}{R} \frac{\partial^2 u}{\partial x^2}, \tag{16}$$

where R is a Reynolds number, with boundary conditions $u(\infty, t) = 0$, $u(-\infty, t) = 1$, is given by

$$u(x, t) = [1 + e^{(1/2)R(x-(1/2)t)}]^{-1}. \tag{17}$$

This solution represents a front of nominal thickness $8/R$ propagating with a positive velocity equal to $\frac{1}{2}$.

This solution forms an ideal test problem for illustrating the properties of the new simplified scheme and for comparing it with the MFE method. First, the solution is monotonic and sufficiently well behaved so that regularization terms were not required. Secondly, as will be shown later, a one-dimensional, single variable equation, such as the Burgers equation, is sufficiently simple in the MFE formulation without regularization terms that the matrix equation (Eq. (5)) can be explicitly solved and put in the form of Eqs. (15). Thus in this case the MFE method may be considered as merely a particular case of SMFE. This clearly simplifies the comparison of the methods.

The numerical solutions for this problem were obtained using a mesh of 20 nodes, $x_i \in [0, 10]$. The two end nodes were fixed while the internal nodes were allowed to move. The boundary conditions at the end points of this region are intended to correspond to the exact solution. Thus, for all cases the boundary conditions applied were: $S_1 = 0$, $u_1 = 1$, $x_1 = 0$; $S_{20} = u_{20} = 0$, $x_{20} = 10$; as well as the implied conditions $\dot{u}_1 = \dot{u}_{20} = 0$. The Reynolds number R was taken to be 10^3, corresponding to a reasonably steep front of nominal thickness 0.008. The initial mesh was distributed uniformly between $x_2 = 0.1$ and $x_{19} = 0.2$. The initial profile was specified to be

$$u_i = \tfrac{1}{2}\{1 + \cos[10\pi(x_i - x_2)]\}, \qquad 2 \leqslant i \leqslant 19.$$

The time integration was performed using a stiff equation solver called SDRIV1 [10]. The general behavior of the numerical solutions is illustrated in Fig. 1. The front starts from the left boundary, very quickly reaches a "steady state," and then propagates with an internal mesh velocity $S_i = 0.5$ toward the right-hand boundary.

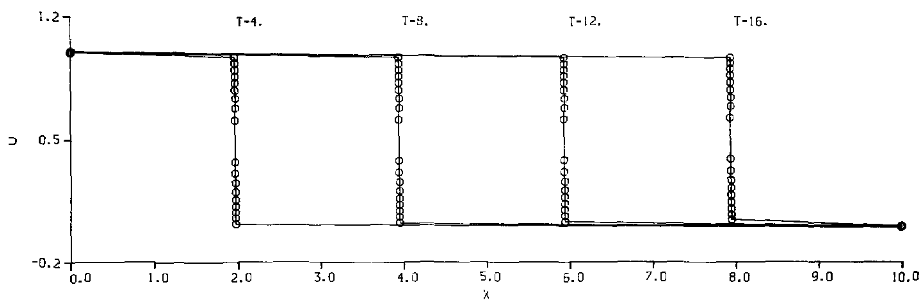

FIG. 1. A typical SMFE solution of the test problem showing the propagation of a Burgers shock front ($R = 10^3$) toward the right-hand boundary.

The numerical solutions were obtained for several cases of SMFE, as well as for the MFE method. The MFE method is the reference for comparison with the other cases. All the cases will be characterized by an approximation to the Burgers equation in the form (cf. Eqs. (2), (12))

$$\dot{u}_i + (u_i - S_i)\left[\frac{\partial u}{\partial x}\right]_i = \frac{1}{R}\left[\frac{\partial^2 u}{\partial x^2}\right]_i, \tag{18}$$

where $[\partial u/\partial x]_i$ and $[\partial^2 u/\partial x^2]_i$ are some suitable approximations to $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ evaluated at point i. A particular choice of these two quantities then determines the matrix equation for the mesh velocities S_i. As mentioned previously, for the case of the one-dimensional Burgers equation, the MFE method without regularization terms can be explicitly solved and expressed as a special case of the SMFE method. This is demonstrated in Appendix B. Thus, the MFE method corresponds to choosing

$$\left[\frac{\partial u}{\partial x}\right]_i = \frac{1}{2}\left[\frac{\partial u^k}{\partial x} + \frac{\partial u^{k-1}}{\partial x}\right], \tag{19}$$

and

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_i = \frac{1}{2}\left[\xi_i + \eta_i\right], \tag{20}$$

where

$$\xi_i = \frac{1}{x_{i+1} - x_i}\left[3\frac{\partial u^k}{\partial x} - 2\frac{\partial u^{k-1}}{\partial x} - \frac{\partial u^{k+1}}{\partial x}\right], \tag{21}$$

$$\eta_i = \frac{1}{x_i - x_{i-1}}\left[2\frac{\partial u^k}{\partial x} - 3\frac{\partial u^{k-1}}{\partial x} + \frac{\partial u^{k-2}}{\partial x}\right], \tag{22}$$

for $2 \leqslant i \leqslant N-1$, $2 \leqslant k \leqslant N-2$, where N is the total number of mesh points. The corresponding matrix equation for the mesh velocities can be solved to give

$$S_i = u_i + \frac{1}{R}\frac{(\eta_i - \xi_i)}{\left[\dfrac{\partial u^k}{\partial x} - \dfrac{\partial u^{k-1}}{\partial x}\right]}. \tag{23}$$

This decomposition of the MFE method will not be possible in general; however, in this case it provides valuable insight into the method and permits direct comparison with other cases of the SMFE method.

The examples of the SMFE method which we will consider will be specified by the following representative choices for $[\partial u/\partial x]_i$ and $[\partial^2 u/\partial x^2]_i$:

1. $$\left[\frac{\partial u}{\partial x}\right]_i = \frac{1}{x_{i+1} - x_{i-1}}\left[(x_{i+1} - x_i)\frac{\partial u^k}{\partial x} + (x_i - x_{i-1})\frac{\partial u^{k-1}}{\partial x}\right]$$

$$= \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}}, \tag{24}$$

i.e., a centered difference approximation,

$$2. \quad \left[\frac{\partial u}{\partial x}\right]_i = \frac{1}{x_{i+1} - x_{i-1}} \left[(x_i - x_{i-1})\frac{\partial u^k}{\partial x} + (x_{i+1} - x_i)\frac{\partial u^{k-1}}{\partial x}\right], \tag{25}$$

$$3. \quad \left[\frac{\partial^2 u}{\partial x^2}\right]_i = \frac{2}{(x_{i+1} - x_{i-1})}\left[\frac{\partial u^k}{\partial x} - \frac{\partial u^{k-1}}{\partial x}\right], \tag{26}$$

both obtained by fitting a quadratic function to u_{i-1}, u_i, and u_{i+1}, and

$$4. \quad \left[\frac{\partial^2 u}{\partial x^2}\right]_i = \frac{1}{2}\frac{(x_{i+1} - x_{i-1})}{(x_{i+1} - x_i)(x_i - x_{i-1})}\left[\frac{\partial u^k}{\partial x} - \frac{\partial u^{k-1}}{\partial x}\right]. \tag{27}$$

This last is an approximation obtained by applying a formula of the type given by Eq. (19) to three values of the derivative: $\partial u^k/\partial x$ and $\partial u^{k-1}/\partial x$, assumed to hold at the midpoints of the intervals k and $k-1$, respectively, and by $\frac{1}{2}(\partial u^k/\partial x + \partial u^{k-1}/\partial x)$, assumed to hold at the point x_i.

The results of the calculations for the test problem are illustrated in Figs. 2–8 for the cases summarized in Table I. The figures show the solution at time $t = 10$ when

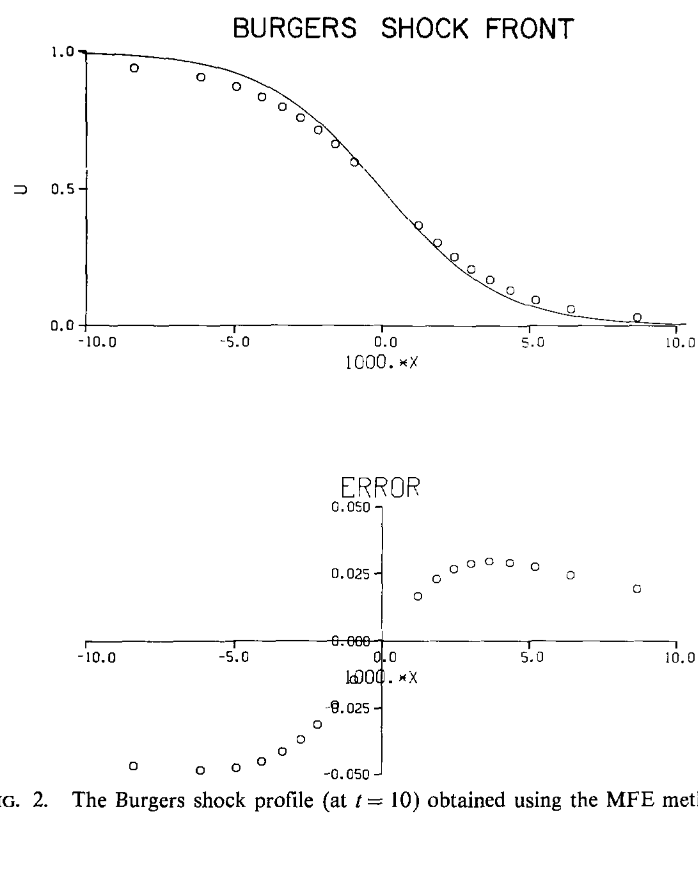

FIG. 2. The Burgers shock profile (at $t = 10$) obtained using the MFE method.

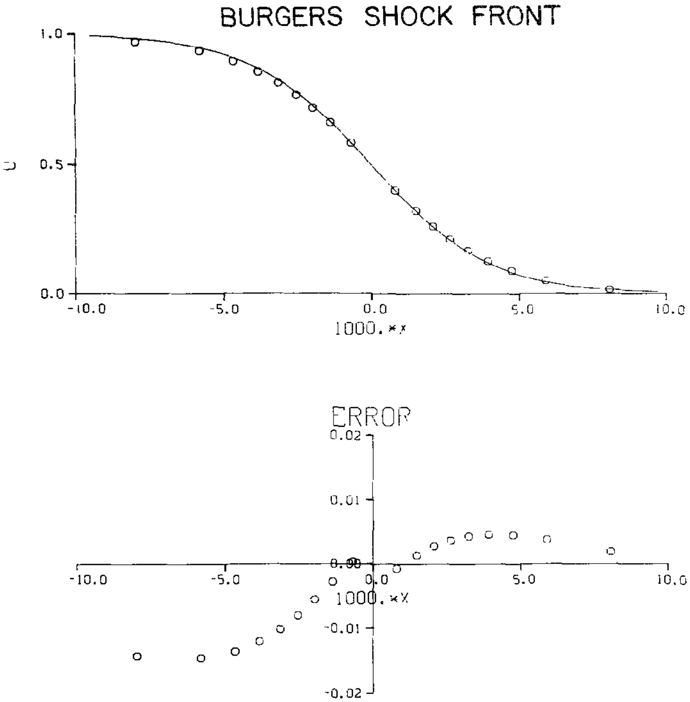

FIG. 3. The Burgers shock profile (at $t = 10$) obtained using the SMFE method and the difference approximations of Eq. (24) and Eq. (26).

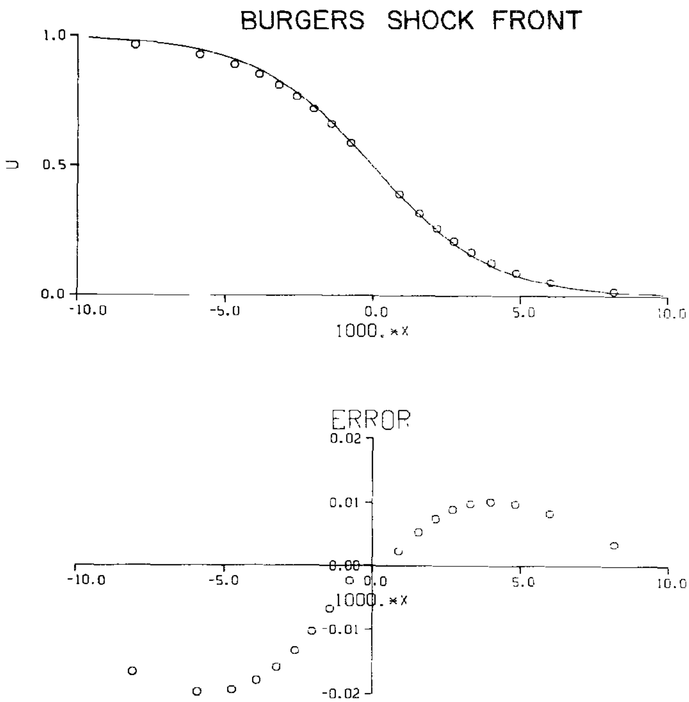

FIG. 4. The Burgers shock profile (at $t = 10$) obtained using the SMFE method and the difference approximations of Eq. (24) and Eq. (27), except for the two nodes at the edges of the profile where Eq. (26) was used.

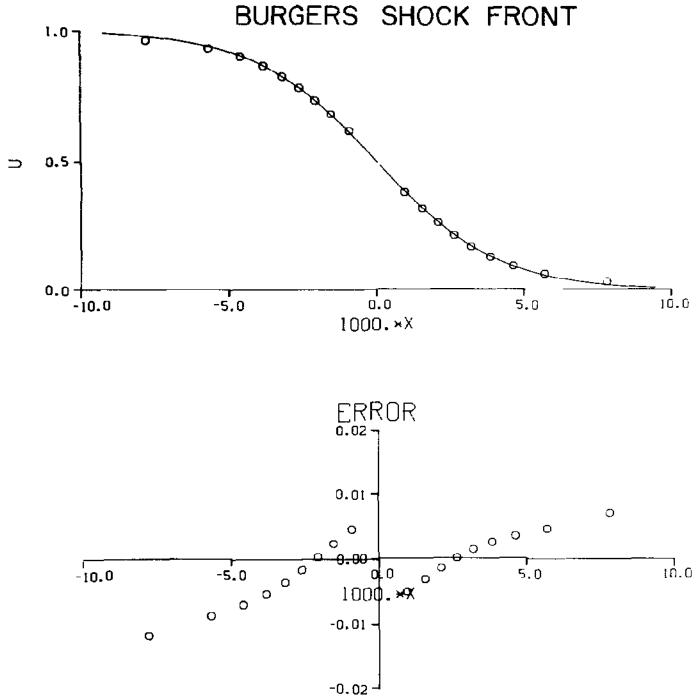

FIG. 5. The Burgers shock profile (at $t = 10$) obtained using the SMFE method and the difference approximations of Eq. (19) and Eq. (26), except for the two nodes at the edges of the profile where Eq. (27) was used.

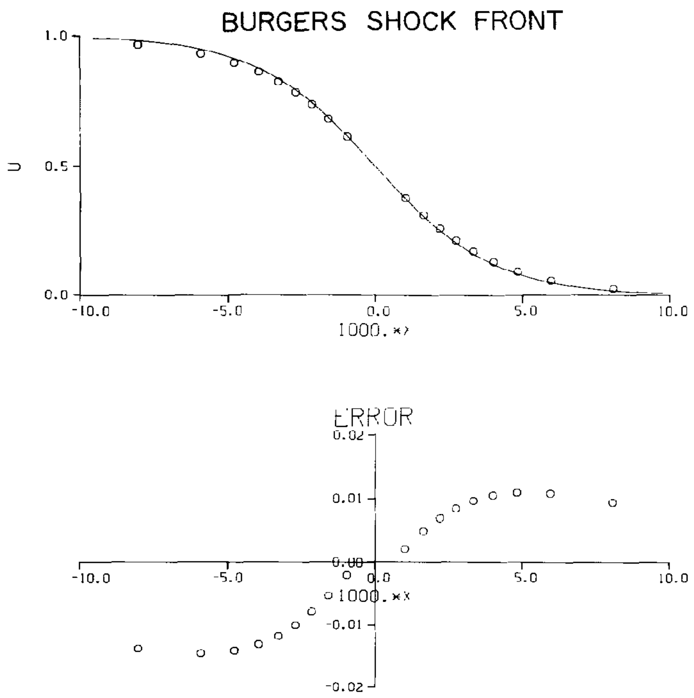

FIG. 6. The Burgers shock profile (at $t = 10$) obtained using the SMFE method and the difference approximations of Eq. (19) and Eq. (27).

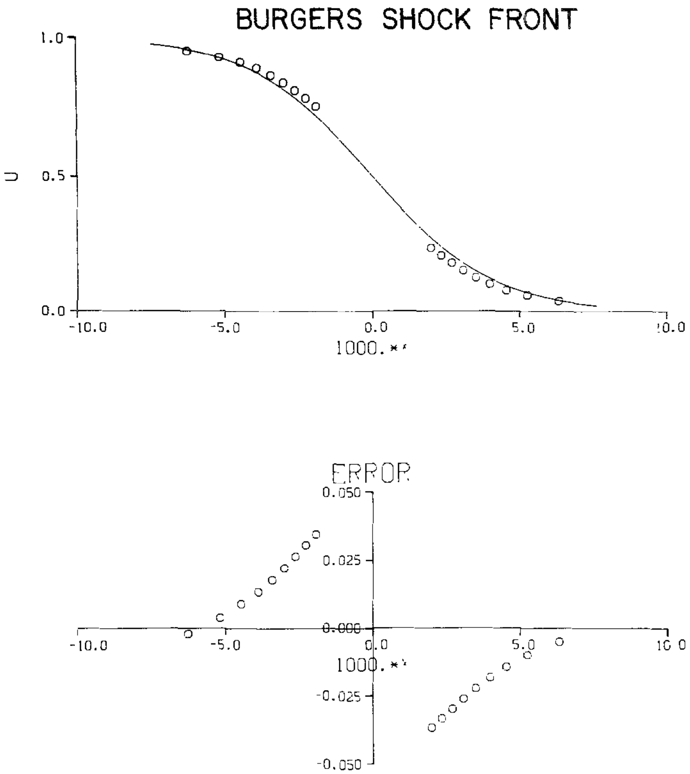

FIG. 7. The Burgers shock profile (at $t = 10$) obtained using the SMFE method and the difference approximations of Eq. (25) and Eq. (26), except for the two nodes at the edges of the profile where Eq. (27) was used.

the front has reached approximately midway between the mesh boundaries. Also plotted for comparison is the exact solution (Eq. (17)) which has been made to coincide with the computed profile at the point where $u = \frac{1}{2}$. The difference between the computed values and the exact solution is the error, and this is plotted separately underneath the plot of the profile. The maximum error for each case is shown in Table I.

A solution could not be obtained for several of the cases. These cases are indicated by asterisks in the table. The difficulty appears to be due to a failure of the approximation (Eq. (18)) at the nodes located at the edges of the profile ($i = 2, 19$) where there is an abrupt change in mesh spacing. By merely switching the approximation $[\partial^2 u/\partial x^2]_i$ (between Eq. (26) and Eq. (27)) at these points alone it was possible to obtain stable solutions, and these are shown in the corresponding figures. Surprisingly, the SMFE method appears to give better accuracy than MFE for our test problem in all cases except possibly for the case of Fig. 7. For this specific case the off-diagonal coefficients of the matrix A_3 of Eq. (14) are proportional to $(\partial u^k/\partial x - \partial u^{k-1}/\partial x)(\partial u^{k+1}/\partial x - \partial u^k/\partial x)$. Each of these factors is proportional to

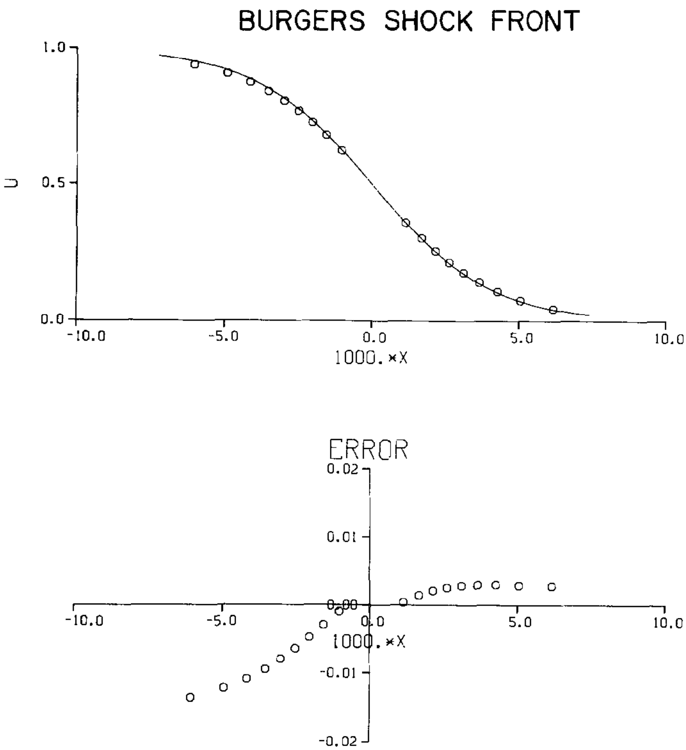

FIG. 8. The Burgers shock profile (at $t = 10$) obtained using the SMFE method and the difference approximations of Eq. (25) and Eq. (27).

$[\partial^2 u/\partial x^2]_i$, and since the second derivative goes through zero within the front profile there is a tendency for mesh points on either side of this point to become decoupled and drift apart. This is apparent in Fig. 7. In general, the large mesh ratio at the end points ($i = 2$, 19) undoubtedly determines the character and accuracy of the solutions. It is important to note that all the solutions were obtained without the use

TABLE I

Summary of Calculations

Figure No.	$[\partial u/\partial x]_i$	$[\partial u^2/\partial x^2]_i$	Maximum Error	Method
2	Eq. (19)	Eq. (20)	0.05	MFE
3	Eq. (24)	Eq. (26)	0.015	SMFE
4*	Eq. (24)	Eq. (27)	0.02	SMFE
5*	Eq. (19)	Eq. (26)	0.012	SMFE
6	Eq. (19)	Eq. (27)	0.015	SMFE
7*	Eq. (25)	Eq. (26)	0.04	SMFE
8	Eq. (25)	Eq. (27)	0.015	SMFE

of regularization terms (in contrast to the calculations in Ref. [7]). Regularization terms were originally introduced at least partly to alleviate similar difficulties, and their judicious use in this problem would undoubtedly improve the accuracy of the solutions. However, it was felt that since the appropriate specification of regularization terms is very subjective and problem dependent, their use would unnecessarily complicate the exposition, as well as the comparison of the MFE and SMFE methods. The significant point is not that SMFE happened to be more accurate than MFE in some of the cases considered, since this will probably not be true for other cases, or other problems, but that SMFE is at least comparable to MFE in its ability to resolve and follow steep fronts when used in a comparable manner.

D. Summary

A rather general technique for simplifying the moving finite element method to obtain an adaptive mesh technique has been described. This method appears to be applicable whenever the MFE method is applicable. The method is derived as a constrained minimization of the MFE residual functional and thus it retains several advantages of the MFE method, such as a symmetric, positive definite matrix, the ability to independently control mesh motion through penalty terms, and a direct relationship to a global measure of solution error.

Many aspects of the method, such as applications to systems of equations or to two-dimensional equations, have not been considered. These questions, however, are entirely similar to the corresponding aspects of the MFE method where, in some cases, they have not yet been adequately dealt with.

Appendix A: Derivation of the SMFE Equations for the Case of the One-Dimensional Burgers Equation

As an example of the application of the SMFE method, we will derive the equations for the case of the one-dimensional Burgers equation (Eq. (16)) in some detail. We start with the definition of the functional (in the absence of regularization terms) as

$$I=\int_{x_1}^{x_N} R^2\,dx=\sum_{k=1}^{N-1}\int_{x_i}^{x_{i+1}} (R^k)^2\,dx, \tag{A1}$$

where

$$R^k=\sum_{j=i}^{i+1}\left[\dot{u}_j+(u_j-S_j)\frac{\partial u^k}{\partial x}\right]\phi_j^k(x)-\frac{1}{R}\frac{\partial^2 u}{\partial x^2}, \tag{A2}$$

and k denotes the subinterval $x^k : x \in [x_i, x_{i+1}]$, $1 \leqslant i \leqslant N$, $1 \leqslant k \leqslant N-1$, and $\phi_j^k(x)$ are the shape functions defined in Eq. (3). We have assumed a linear variation for all variables; this leaves the functional dependence of $\partial^2 u/\partial x^2$ undefined for the moment.

The SMFE method is specified by a choice of the discrete approximation in the form

$$\dot{u}_i = (S_i - u_i)\left[\frac{\partial u}{\partial x}\right]_i + \frac{1}{R}\left[\frac{\partial^2 u}{\partial x^2}\right]_i, \tag{A3}$$

where $[\partial u/\partial x]_i$, $[\partial^2 u/\partial x^2]_i$ are suitable approximations. Substituting this into the equation for the residual, we obtain

$$\begin{aligned} R^k = {} & (S_i - u_i)\left(\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^k}{\partial x}\right)\phi_i^k(x) \\ & + (S_{i+1} - u_{i+1})\left(\left[\frac{\partial u}{\partial x}\right]_{i+1} - \frac{\partial u^k}{\partial x}\right)\phi_{i+1}^k(x) \\ & + \frac{1}{R}\left\{\left[\frac{\partial^2 u}{\partial x^2}\right]_i \phi_i^k(x) + \left[\frac{\partial^2 u}{\partial x^2}\right]_{i+1}\phi_{i+1}^k(x) - \frac{\partial^2 u}{\partial x^2}\right\}. \end{aligned} \tag{A4}$$

This incorporates the constraint into the functional. Thus, the functional I depends only on the set of unknowns $\{S_i\}$. A typical equation to determine the unknowns is obtained when we minimize the functional with respect to S_i. From Eq. (A4) we see that only two terms in the functional depend on S_i, namely, those involving R^{k-1} and R^k. Thus, the equation associated with point i is

$$\begin{aligned} \frac{\partial I}{\partial S_i} &= \frac{\partial}{\partial S_i}\left[\int_{x_{i-1}}^{x_i} (R^{k-1})^2\, dx + \int_{x_i}^{x_{i+1}} (R^k)^2\, dx\right] \\ &= 2\left[\int_{x_{i-1}}^{x_i} R^{k-1}\frac{\partial R^{k-1}}{\partial S_i}\, dx + \int_{x_i}^{x_{i+1}} R^k \frac{\partial R^k}{\partial S_i}\, dx\right] \\ &= 0, \end{aligned} \tag{A5}$$

where, by differentiating Eq. (A4), we have

$$\begin{aligned} \frac{\partial R^{k-1}}{\partial S_i} &= \left(\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^{k-1}}{\partial x}\right)\phi_i^{k-1}(x), \\ \frac{\partial R^k}{\partial S_i} &= \left(\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^k}{\partial x}\right)\phi_i^k(x). \end{aligned} \tag{A6}$$

Equation (A5) corresponds to the assembly process in a typical finite-element procedure. Examining these equations, we see that we will need the integrals

$$\int_{x_i}^{x_{i+1}} [\phi_i^k(x)]^2\, dx = \int_{x_i}^{x_{i+1}} [\phi_{i+1}^k(x)]^2\, dx = (x_{i+1} - x_i)/3, \tag{A7}$$

and

$$\int_{x_i}^{x_{i+1}} \phi_i^k(x)\, \phi_{i+1}^k(x)\, dx = (x_{i+1} - x_i)/6, \tag{A8}$$

as well as

$$\int_{x_i}^{x_{i+1}} \phi_i^k(x) \frac{\partial^2 u}{\partial x^2}\, dx \qquad \text{and} \qquad \int_{x_i}^{x_{i+1}} \phi_{i+1}^k(x) \frac{\partial^2 u}{\partial x^2}\, dx.$$

These last two integrals will be evaluated, as in the MFE method, by the process of "mollification" [7]. That is, since we assume a linear variation of u over each interval there will be a discontinuity in $\partial u/\partial x$ at each mesh point. These discontinuities are smoothed over a short distance δ, the integrals are evaluated, and the result is obtained by taking the limit $\delta \to 0$. The result is

$$\int_{x_i}^{x_{i+1}} \phi_i^k(x) \frac{\partial^2 u}{\partial x^2}\, dx = \frac{1}{2}\left(\frac{\partial u^k}{\partial x} - \frac{\partial u^{k-1}}{\partial x}\right), \tag{A9}$$

and

$$\int_{x_i}^{x_{i+1}} \phi_{i+1}^k(x) \frac{\partial^2 u}{\partial x^2}\, dx = \frac{1}{2}\left(\frac{\partial u^{k+1}}{\partial x} - \frac{\partial u^k}{\partial x}\right). \tag{A10}$$

Finally, by using these results and taking out common factors, Eq. (A5) may be expressed as

$$A_i(S_{i-1} - u_{i-1}) + B_i(S_i - u_i) + C_i(S_{i+1} - u_{i+1}) = D_i, \tag{A11}$$

where

$$\begin{aligned}
A_i &= (x_i - x_{i-1}) \left(\left[\frac{\partial u}{\partial x}\right]_{i-1} - \frac{\partial u^{k-1}}{\partial x} \right) \left(\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^{k-1}}{\partial x} \right), \\
B_i &= 2\left[(x_i - x_{i-1}) \left(\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^{k-1}}{\partial x} \right)^2 + (x_{i+1} - x_i) \left(\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^k}{\partial x} \right)^2 \right], \\
C_i &= (x_{i+1} - x_i) \left(\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^k}{\partial x} \right) \left(\left[\frac{\partial u}{\partial x}\right]_{i+1} - \frac{\partial u^k}{\partial x} \right), \\
D_i &= \frac{3}{R} \left(\frac{\partial u^k}{\partial x} - \frac{\partial u^{k-1}}{\partial x} \right) \left(2\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^k}{\partial x} - \frac{\partial u^{k-1}}{\partial x} \right) \\
&\quad - \frac{1}{R}(x_i - x_{i-1}) \left(\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^{k-1}}{\partial x} \right) \left(\left[\frac{\partial^2 u}{\partial x^2}\right]_{i-1} + 2\left[\frac{\partial^2 u}{\partial x^2}\right]_i \right) \\
&\quad - \frac{1}{R}(x_{i+1} - x_i) \left(\left[\frac{\partial u}{\partial x}\right]_i - \frac{\partial u^k}{\partial x} \right) \left(\left[\frac{\partial^2 u}{\partial x^2}\right]_{i+1} + 2\left[\frac{\partial^2 u}{\partial x^2}\right]_i \right).
\end{aligned} \tag{A12}$$

These equations hold everywhere except at the two points next to the ends of the interval ($i=2$, $N-1$). These two equations are modified because the residuals in the two subintervals, R^1 and R^{N-1}, are affected by the boundary conditions $S_1=0$, $u_1=1$, and $S_N=0$, $u_N=0$. These end equations are easily obtained by using the above boundary conditions in Eq. (A11), together with $[\partial u/\partial x]_i=[\partial^2 u/\partial x^2]_i=0$, obtained from the implied conditions $\dot{u}_i=0$, $i=1,N$. Thus, Eq. (A11) represents a tridiagonal system of equations which is easily solved for the mesh velocities $\{S_i\}$.

APPENDIX B: EXPLICIT SOLUTION OF THE MFE EQUATIONS FOR THE ONE-DIMENSIONAL BURGERS EQUATION

The one-dimensional Burgers equation belongs to a class of equations for which the MFE equations, without regularization terms, may be explicitly solved.

The MFE method is associated with the functional

$$I=\sum_k \int_{x_i}^{x_{i+1}} (R^k)^2\,dx, \tag{A1}$$

where, for the Burgers equation,

$$R^k=\sum_{j=i}^{i+1}\left[\dot{u}_j+(u_j-S_j)\frac{\partial u^k}{\partial x}\right]\phi_j^k(x)-\frac{1}{R}\frac{\partial^2 u}{\partial x^2}, \tag{A2}$$

and $\dot{u}_i$, S_i, $1\leqslant i\leqslant N$, are the variables. We now introduce new variables $\{\xi_i,\eta_i\}$ by means of

$$\frac{1}{R}\xi_i=\dot{u}_i+(u_i-S_i)\frac{\partial u^k}{\partial x}, \tag{B1}$$

$$\frac{1}{R}\eta_i=\dot{u}_i+(u_i-S_i)\frac{\partial u^{k-1}}{\partial x}, \tag{B2}$$

which can be inverted to give

$$\dot{u}_i=\frac{1}{R}\frac{\left(\eta_i\dfrac{\partial u^k}{\partial x}-\xi_i\dfrac{\partial u^{k-1}}{\partial x}\right)}{\left(\dfrac{\partial u^k}{\partial x}-\dfrac{\partial u^{k-1}}{\partial x}\right)}, \tag{B3}$$

and

$$S_i=u_i+\frac{1}{R}\frac{(\eta_i-\xi_i)}{\left(\dfrac{\partial u^k}{\partial x}-\dfrac{\partial u^{k-1}}{\partial x}\right)}. \tag{B4}$$

Clearly, there is a one-to-one correspondence between the variables $\{\dot{u}_i, S_i\}$ and $\{\xi_i, \eta_i\}$, provided $\partial u^k/\partial x \neq \partial u^{k-1}/\partial x$. If $\partial u^k/\partial x = \partial u^{k-1}/\partial x$ then the method becomes singular, as discussed previously, and MFE breaks down. Hence, in the absence of this singularity we can perform the minimization with respect to the new variables $\{\xi_i, \eta_i\}$ just as well as with respect to the old variables $\{\dot{u}_i, S_i\}$. Thus, the minimum is found by solving

$$\frac{\partial I}{\partial \xi_i} = \frac{2}{R} \int_{x_i}^{x_{i+1}} \phi_i^k(x)\, R^k\, dx = 0, \tag{B5}$$

$$\frac{\partial I}{\partial \eta_{i+1}} = \frac{2}{R} \int_{x_i}^{x_{i+1}} \phi_{i+1}^k(x)\, R^k\, dx = 0, \tag{B6}$$

where

$$R^k = \frac{1}{R}\left[\phi_i^k(x)\, \xi_i + \phi_{i+1}^k(x)\, \eta_{i+1} - \frac{\partial^2 u}{\partial x^2}\right]. \tag{B7}$$

Evaluating the integrals with the help of Eqs. (A7)–(A10), we obtain

$$2\xi_i + \eta_{i+1} = \frac{3}{(x_{i+1} - x_i)} \left(\frac{\partial u^k}{\partial x} - \frac{\partial u^{k-1}}{\partial x}\right), \tag{B8}$$

$$\xi_i + 2\eta_{i+1} = \frac{3}{(x_{i+1} - x_i)} \left(\frac{\partial u^{k+1}}{\partial x} - \frac{\partial u^k}{\partial x}\right), \tag{B9}$$

which can be solved to give

$$\xi_i = \frac{1}{(x_{i+1} - x_i)} \left[3\frac{\partial u^k}{\partial x} - 2\frac{\partial u^{k-1}}{\partial x} - \frac{\partial u^{k+1}}{\partial x}\right], \tag{B10}$$

$$\eta_{i+1} = \frac{1}{(x_{i+1} - x_i)} \left[2\frac{\partial u^{k+1}}{\partial x} + \frac{\partial u^{k-1}}{\partial x} - 3\frac{\partial u^k}{\partial x}\right], \tag{B11}$$

or, equivalently

$$\eta_i = \frac{1}{(x_i - x_{i-1})} \left[2\frac{\partial u^k}{\partial x} + \frac{\partial u^{k-2}}{\partial x} - 3\frac{\partial u^{k-1}}{\partial x}\right]. \tag{B12}$$

These values of ξ_i, η_i may be used directly in Eqs. (B3) and (B4), but it is more interesting to express Eq. (B3) in an alternative form obtained by taking the average of Eqs. (B1) and (B2),

$$\dot{u}_i = \frac{1}{2}\left[\frac{\partial u^k}{\partial x} + \frac{\partial u^{k-2}}{\partial x}\right](S_i - u_i) + \frac{1}{2R}(\xi_i + \eta_i), \tag{B13}$$

which is suggestive of an approximation to the Burgers equation.

The above expressions for ξ_i, η_i apply in the interior of the mesh. Applying the boundary conditions appropriate to the test problem ($S_1 = S_N = 0$, $u_1 = 1$, $u_N = 0$, $\dot{u}_1 = \dot{u}_N = 0$) to the residuals in the end-intervals results in

$$\xi_1 = R \frac{\partial u^1}{\partial x}, \qquad \eta_1 = 0,$$
$$\eta_2 = \frac{3}{2} \frac{1}{(x_2 - x_1)} \left(\frac{\partial u^2}{\partial x} - \frac{\partial u^1}{\partial x} \right) - \frac{1}{2} \xi_1, \tag{B14}$$

and

$$\xi_N = \eta_N = 0,$$
$$\xi_{N-1} = \frac{3}{2} \frac{1}{(x_N - x_{N-1})} \left(\frac{\partial u^{N-1}}{\partial x} - \frac{\partial u^{N-2}}{\partial x} \right). \tag{B15}$$

Thus, although Eqs. (B10) and (B12) involve five point operators in the interior, no outside points are ever required.

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